# MORSE HOMOLOGY: EXAM PREPARATION

IPSITA DATTA

(1) Morse Lemma (Statement and Proof)
(2) Section 1.4 of Audin–Damian: Examples of Morse Functions
(3) Suppose the $f : M \to \mathbb{R}$ is a Morse function on a compact manifold $M$, and $X$ is a pseudogradient of $f$. Let $\gamma : \mathbb{R} \to M$ be a trajectory of $X$. Show that there exist critical points $c$ and $d$ of $f$ such that

$$\lim_{s\to-\infty} \gamma(s) = c \text{ and } \lim_{s\to\infty} \gamma(s) = d.$$

(4) Show that for a Morse function $f : M \to \mathbb{R}$, if $a$ and $b$ are real numbers such that $f$ does not have any critical value in the interval $[a, b]$ and $f^{-1}([a, b])$ is compact, then the sublevel sets $M^a$ and $M^b$ are diffeomorphic.
(5) Reeb's Theorem (Statement and Proof).
(6) Compute Morse homology directly from a Morse function of
    - Spheres $S^2$ (two different Morse functions), $S^n$,
    - Torus $T^2$,
    - $\mathbb{CP}^n$,
    - Unit disk.
(7) Morse Inequalities (Statement and Proof).
(8) Can there be a Morse function on the torus $T^2$ with exactly one critical point of index 1?
(9) Brouwer Fixed Point theorem (Statement and proof using Morse homology).
(10) Künneth Formula (Statement and Proof).
(11) Compute the Morse homology of $\mathbb{RP}^n$.
(12) Suppose $X$ and $Y$ are two smooth manifolds. Show that Euler characteristic is multiplicative, that is,

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

(13) Compute Euler characteristic of $S^n$ for $n \geq 1$, and $T^n$ for $n \geq 1$.
(14) Show that if $M$ is a simply connected compact smooth manifold, then $HM_1(M; \mathbb{Z}/2\mathbb{Z}) = 0$.
(15) Show that if a manifold $M$ admits a Morse function with no critical points of index 1, then it is simply connected.
(16) Show that there is no retraction from $S^n$ to a subsphere $S^k \subset S^n$ for $k < n$.
(17) Show that if $M$ is a submanifold of a manifold $W$ and if $U$ is a tubular neighbourhood of $M$ in $W$, then $HM_*(U) \cong HM_*(M)$.
(18) For $a \in (D^N)^\circ$, the interior of the unit disk of dimension $N$, and a smooth manifold $M$, let $i_a : M \to D^N \times M$ be the embedding defined by $i_a(x) = (a, x)$. Then
    - $(i_0)_*$ is an isomorphism,
    - $(i_a)_* = (i_0)_*$.
(19) Exercises 2, 3, 4, 5, 11, 12, 16, 17, 20, 21, 22, 23, from Audin–Damian.
(20) What is your favourite theorem/fact/example from the class? Why?